Exercise Sheet 4: Cofiber Sequences

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Contents

1 Instructions

Please complete all exercises on this sheet. There are two exercises in the second section and seven in the third section. Several of the exercises can be rigorously answered by a paragraph of text. There are no exercises in the last section.

2 Pointed Cofibrations

When we introduced cofibrations the first time we worked exclusively in the unbased category. In this exercise we shall study the analogues in the pointed category. Past formalising the definitions and observing a few subtle points there is not much we need to dwell on.

Definition 1 *A pointed map $j : A \to X$ is a **pointed cofibration** if it satisfies the pointed homotopy extension property with respect to all based spaces. That is, given the solid part of any strictly commuting diagram in Top_**

$$\begin{tikzcd}
A \arrow[r, hook, "in_0"] \arrow[d, "j"'] & A \wedge I_+ \arrow[d, "j\wedge 1"'] \arrow[ddr, bend left, "F"] & \\
X \arrow[r, "in_0"] \arrow[drr, bend right, "f"'] & X \wedge I_+ \arrow[dr, dashed, "\widetilde{F}"] & \\
& & Y.
\end{tikzcd} \tag{2.1}$$

the dotted extension can be filled in. $\square$

Recall that $X \wedge I_+ \cong (X \times I)/(* \times I)$, so the diagram is just a way of enforcing that pointed homotopies are used throughout. The main results go through in the based category as in the unbased category. We recall below some of the most salient points. We'll call a cofibration as defined in 1 a pointed, or based, cofibration, and sometimes say that it is a cofibration in Top_*. This is in contrast to the definition given in exercise sheet 2 in the unbased category. We'll call the cofibrations defined there unpointed, unbased, or free, and say that they are cofibrations in Top. In light of Exercise 1 below, we can often get away with being less pedantic.

We define the **pointed mapping cylinder** M_j of a pointed map $j : A \to X$ by means of the pushout in Top_*

$$\begin{tikzcd} A \arrow[r, "in_0"] \arrow[d, "j"'] & A \wedge I_+ \arrow[d] \\ X \arrow[r] & M_j \end{tikzcd} \qquad M_j = \frac{X \vee (A \wedge I_+)}{[j(a) \sim (a,0)]}. \tag{2.2}$$

Then the following is familiar.

Proposition 2.1 *A map $j : A \to X$ is a pointed cofibration if and only if the pointed mapping cylinder M_j is a retract of $X \wedge I_+$.* ∎

The same proof as in exerise sheet 2 can now be used to show that a pointed cofibration is necessarily an embedding (it need not be closed).

Example 2.1

1. *If $i : A \hookrightarrow B$ and $j : B \hookrightarrow C$ are cofibrations in Top_*, then so is $ji : A \hookrightarrow C$.*
2. *If $i : A \hookrightarrow X$ and $j : B \hookrightarrow Y$ are cofibrations in Top_*, then so is $i \vee j : A \vee B \hookrightarrow X \vee Y$.*
3. *A pushout in Top_* of a pointed cofibration is a pointed cofibration.* □

The perhaps unexpected point is that it is actually easier for a map to be a pointed cofibration than for it to be a free cofibration. That is, there are more pointed cofibrations than unpointed cofibrations.

Exercise 2.1 Assume that $j : A \to X$ is a based map. Show that if j is an unpointed cofibration, then it is a pointed cofibration. (Hint: correctly identify the adjoint form of the defining diagram (2.1).) □

Exercise 2.2 Give an example of a map which is a cofibration in Top_* but not in Top. (Hint: $A = *$.) □

Here are some more simple examples of maps which are pointed cofibrations.

Example 2.2 *If X is any pointed space, then the inclusion $j : X \hookrightarrow X \wedge I_+$ is a pointed cofibration. Notice that*

$$(X \wedge I_+) \wedge I_+ \cong X \wedge (I_+ \wedge I_+) \cong X \wedge (I \times I)_+ \tag{2.3}$$

whilst we can identify

$$M_j \cong \frac{(X \wedge I_+) \vee (X \wedge I_+)}{[(x,0) \sim (x,0)]} \cong X \wedge (I \times 0 \cup 0 \times I)_+. \tag{2.4}$$

Choosing a retraction $r : I \times I \to I \times 0 \cup 0 \times I$ *we get a retraction*

$$1 \wedge r : (X \wedge I_+) \wedge I_+ \to M_j. \tag{2.5}$$

(If you have difficulty visualising this consider first the simple case $X = S^0$*).* $\square$

Example 2.3 *If* $j : A \hookrightarrow X$ *is a pointed cofibration and* K *is locally compact, then* $j \wedge 1 : A \wedge K \hookrightarrow X \wedge K$ *is a pointed cofibration. This is because the left adjoint functor* $(-) \wedge K$ *preserves pushouts*[1]*, so the square*

$$\begin{tikzcd} A \wedge K \arrow[r, "in_0"] \arrow[d, "j\wedge 1"'] \arrow[dr, phantom, "\lrcorner"] & (A \wedge K) \wedge I_+ \arrow[d] \\ X \wedge K \arrow[r] & M_j \wedge K \end{tikzcd} \tag{2.6}$$

gives a homeomorphism $M_{j\wedge 1} \cong M_j \wedge K$*. Thus a retraction* $X \wedge I_+ \to M_j$ *for* j *induces one* $X \wedge K \wedge I_+ \to M_{j\wedge 1}$ *also for* $j \wedge 1$*.* $\square$

Example 2.4 *If* $j : A \hookrightarrow X$ *is a pointed cofibration, then so is*

$$\Sigma j : \Sigma A \hookrightarrow \Sigma X. \tag{2.7}$$

This follows directly from example 2.3. $\square$

Example 2.5 *If* X *is any space, then the inclusion*

$$j : X \hookrightarrow CA \tag{2.8}$$

into its reduced cone is a pointed cofibration. This can be seen similarly to example 2.2, since there is a homeomorphism $M_j \cong X \wedge I$*. If* X *is locally compact, then we can also just appeal to example 2.3, since* J *is the map* $X \wedge S^0 \hookrightarrow X \wedge I$*.* $\square$

The examples help illustrate the idea that the theories of pointed and unpointed coifbrations are essentially the same. There are some nasty difficulties that crop up in the pointed theory, however. Occasionally some of the maps you might want to be pointed cofibrations fail to be so. Other times maps which are pointed cofibrations fail to be unpointed cofibrations. For instance inclusions such as

$$X \hookrightarrow X \vee Y, \qquad X \vee Y \hookrightarrow X \times Y \tag{2.9}$$

will not be unpointed cofibrations unless the basepoints in X, Y have good local properties (regardless of whether or not they are pointed cofibrations). To save difficulties we often restrict ourselves to considering spaces with sufficiently nice basepoints.

[1] See *The Category of Pointed Topological Spaces* Proposition 1.2 and pg. 8

Definition 2 *A based space X is said to be **almost well-pointed**, or **nearly well-pointed** if $* \hookrightarrow X$ is a cofibration in Top. It is said to be **well-pointed** if $* \hookrightarrow X$ is a closed cofibration in Top.* $\square$

Nice spaces such as CW complexes and manifolds are well-pointed for any choice of basepoint. The following important theorem allows us to transfer information between the pointed and unpointed categories. The proof is quite technical and is due to Strøm [2] pg. 440. A textbook account can be found in [1] pg. 13.

Theorem 2.2 *Let $j : A \hookrightarrow X$ be a pointed inclusion. If both A and X are well-pointed, then j is a pointed cofibration if and only if it is an unpointed cofibration.* ▮

3 Cofiber Sequences

In this section we will work exclusively with based space. Thus all spaces, maps and homotopies are pointed, and *cofibration* means pointed cofibration.

We begin with an auxiliary notion. Let

$$R \xrightarrow{f} S \xrightarrow{g} T \tag{3.1}$$

be a sequence of pointed sets and pointed functions. We call the sequence **exact** if

$$f(R) = g^{-1}(*). \tag{3.2}$$

The set $g^{-1}(*) \subseteq S$ is often called the **kernel** of g. Note that this is a very weak property compared to, say, exactness for sequences of groups. Although the maps f, g in (3.1) are well-behaved around the basepoint, they can have arbitrary behaviour away from it. For instance, although exactness of

$$S \xrightarrow{g} T \to 0 \tag{3.3}$$

implies that g is surjective, exactness of

$$0 \to R \xrightarrow{f} S \tag{3.4}$$

does not imply that f is injective[2]!

We extend exactness to arbitrary long sequences of pointed sets by saying that

$$\dots S_{n+2} \to S_{n+1} \to S_n \to S_{n-1} \to S_{n-2} \to \dots \tag{3.5}$$

is **exact** if each three-term subsequence $S_{n+1} \to S_n \to S_{n-1}$ is exact in the sense of (3.1).

Now let $j : A \hookrightarrow X$ be a cofibration. Then A is an embedded subspace of X and we can form the quotient space X/A. Denote by $q : X \to X/A$ the quotient map. We call the sequence of spaces and maps

$$A \xrightarrow{j} X \xrightarrow{q} X/A \tag{3.6}$$

a strict **cofiber sequence**. Although this sequence is exact in the previous sense, it is not the reason for having introduced it. Rather this cofiber sequence gives us exact sequences.

[2] Make sure you understand this!

Exercise 3.1 We keep the notation of the last paragraph. Let Y be any space. Given a map $f : X \to Y$, show that $fj \simeq *$ if and only if there is a map $g : X \to Y$ such that $f \simeq g$ and $g(A) = *$. Conclude that the sequence of pointed sets

$$[A, Y] \xleftarrow{j^*} [X, Y] \xleftarrow{q^*} [X/A, Y] \tag{3.7}$$

is exact. Give specific examples to show that j^* need not be surjective and that q^* need not be injective[3].□

Recall that if $j : A \hookrightarrow X$ is a cofibration, then so is $\Sigma j : \Sigma A \hookrightarrow \Sigma X$, and moreover $\Sigma X/\Sigma A \cong \Sigma(X/A)$, since Σ preserves pushouts in Top_*. Thus applying the suspension functor Σ to the cofiber sequence 3.6, yields a new cofiber sequence

$$\Sigma A \xrightarrow{\Sigma j} \Sigma X \xrightarrow{\Sigma q} \Sigma X/\Sigma A. \tag{3.8}$$

Now, if Y is any space, then according to Exercise 3.1

$$[\Sigma A, Y] \xleftarrow{\Sigma j^*} [\Sigma X, Y] \xleftarrow{\Sigma q^*} [\Sigma X/\Sigma A, Y] \tag{3.9}$$

is an exact sequence of pointed sets. On the other hand, each suspension is a grouplike co-H-space, so in this case we can find more structure in this sequence.

Exercise 3.2 Show that (3.7) is an exact sequence of groups[4] and homomorphisms, and that

$$[\Sigma^2 A, Y] \xleftarrow{\Sigma^2 j^*} [\Sigma^2 X, Y] \xleftarrow{\Sigma^2 q^*} [\Sigma^2 X/\Sigma^2 A, Y] \tag{3.10}$$

is an exact sequence of abelian groups and homomorphisms. □

We would like next, if possible, to extend the sequence (3.7) to the right, preferably incorporating some of the extra structure seen in (3.9), (3.10). What's not quite clear is exactly how this should be accomplished. If (3.7) is to extend to the right, then maybe so too should the cofiber sequence (3.6) from which it was derived? But while j is an inclusion, $q : X \to X/A$ is surjective, so there is no way to form another interesting quotient space on the right of (3.6).

Here's the trick we'll use. The cofiber sequence (3.6) actually has more rigidity than we need. In particular $qj = *$ holds strictly, when really we would only need this equation to hold up to homotopy.

Definition 3 *Let $j : A \to X$ be any map. We define the reduced* ***mapping cone*** *C_j of j by means of the pushout diagram*

$$\begin{tikzcd} A \arrow[r, "in_0"] \arrow[d, "j"'] & CA \arrow[d] \\ X \arrow[r, "i_j"'] & C_j \end{tikzcd} \qquad\qquad C_j = \frac{X \vee CA}{[j(a) \sim [a, 0]]}. \tag{3.11}$$

We use the notation $i_j : X \to C_j$ for the canonical inclusion. □

[3]You may use the (as yet unproven) fact that $\pi_n S^n \cong \mathbb{Z}$, $n \geq 1$.

[4]Note that I did not say *short* exact.

Now the composite $i_j \circ j$ is not identically constant, but rather comes equipped with a *canonical* null homotopy. In fact C_j is the universal space with this property, for using the universal property of the pushout we see that a map $C_j \to Y$ is *exactly* a pair of a map $f : X \to Y$ and a null homotopy $F : fj \simeq *$. We will return to the mapping cone construction in future and understand better where it comes from. For now it will suffice to generate some interesting mathematics for us.

For the next two exercises the reader may use results from Exercise Sheet 2: Cofibrations. The reader should assume that any result in §2 and §3 of that sheet is also available in the pointed category, and may quote anything there freely.

Exercise 3.3 Let $j : A \to X$ be an arbitrary map. Show that $i_j : X \hookrightarrow C_j$ is a cofibration and $C_j/X \cong \Sigma A$. $\square$

Thus if $j : A \to X$ is any map at all, then

$$X \xrightarrow{i_j} C_j \xrightarrow{q_j} \Sigma A \tag{3.12}$$

is a cofiber sequence, where $q_j : C_j \to C_j/X \cong \Sigma A$ is the quotient. In particular, if Y is any space, then

$$[X, Y] \xleftarrow{i_j^*} [C_j, Y] \xleftarrow{q_j^*} [\Sigma A, Y] \tag{3.13}$$

is an exact sequence of pointed sets.

Exercise 3.4 Now assume that $j : A \hookrightarrow X$ is a cofibration. Return to the square (3.11) and show that in this case the map $CA \hookrightarrow C_j$ is a cofibration and there is a homeomorphism $C_j/CA \cong X/A$. Show that the quotient map $C_j \to C_j/CA \cong X/A$ is a homotopy equivalence. $\square$

Thus under the assumptions of the last exercise there is a strictly commutative diagram

$$\begin{tikzcd} & X \arrow[r, "i_j"] \arrow[d, equal] & C_j \arrow[r, "q_j"] \arrow[d, "\simeq"'] & \Sigma A \\ A \arrow[r, "j"] & X \arrow[r, "q"] & X/A & \end{tikzcd} \tag{3.14}$$

in which the vertical map $C_j \to C_j/CA \cong X/A$ is a homotopy equivalence, and the three terms in each row each form a cofiber sequence.

Exercise 3.5 Assume that $j : A \hookrightarrow X$ is a cofibration and let Y be a space. Show that there is an exact sequence of pointed sets

$$[A, Y] \xleftarrow{j^*} [X, Y] \xleftarrow{q^*} [X/A, Y] \xleftarrow{\delta^*} [\Sigma A, Y]. \tag{3.15}$$

The map δ^* here is induced by a map $\delta : X/A \to \Sigma A$. What is it? $\square$

Having come this far we may as well continue. We form the mapping cone C_{i_j} of the map $i_j : X \to C_j$

$$\begin{tikzcd} X \arrow[r, "in_0"] \arrow[d, "i_j"'] & CX \arrow[d] \\ C_j \arrow[r] \arrow[ur, phantom, "\lrcorner"] & C_{i_j}. \end{tikzcd} \tag{3.16}$$

and iterate the construction. The map i_j in this square is a cofibration, so everything in exercises 3.3 and 3.4 is applicable. In particular the quotient map

$$C_{i_j} \to C_{i_j}/CX \cong C_j/X \cong \Sigma A \tag{3.17}$$

is a homotopy equivalence, and by identifying $C_{i_j}/C_j \cong \Sigma X$ we get a cofiber sequence

$$C_j \xrightarrow{i_{i_j}} C_{i_j} \xrightarrow{q_{i_j}} \Sigma X. \tag{3.18}$$

This is related to (3.12) through the diagram

$$\begin{tikzcd} & & C_j \arrow[r, "i_{i_j}"] \arrow[d, equal] & C_{i_j} \arrow[r, "q_{i_j}"] \arrow[d, "\simeq"'] & \Sigma X \\ & X \arrow[r, "i_j"] \arrow[d, equal] & C_j \arrow[r, "q_j"] \arrow[d, "\simeq"'] & \Sigma A \arrow[ur, dashed] & \\ A \arrow[r, "j"] & X \arrow[r, "q"] & X/A. \arrow[ur, dashed, "\delta"'] & & \end{tikzcd} \tag{3.19}$$

Exercise 3.6 Show that the composite $\Sigma A \simeq C_{i_j} \xrightarrow{q_{i_j}} \Sigma X$ indicated by the unlabeled dotted arrow is homotopic to the map $-\Sigma j$. (Hint: to get the sign right you need to think about how the coordinates of the cone and the suspension get mixed up by the homotopy equivalence. You can form a geometric picture by thinking of each successive cone as being attaching *on the top* of the space. This necessarily changes the orientation of the last attached cone, and the -1 sign appears as a consequence. Your work in Exercise 2.1 of *Co-H-Spaces* may be useful to you in this problem.) $\square$

Next we replace q_{i_j} up to homotopy by the inclusion of C_{i_j} into the mapping cone of i_{i_j}. This last space is homotopy equivalent to $\Sigma X/\Sigma A$, and we can check that the map $\Sigma X \to \Sigma X/\Sigma A$ which is supplied by our construction is homotopic to $-\Sigma q$.

Continuing on we arrive at the **long cofiber sequence**

$$A \xrightarrow{j} X \xrightarrow{q} X/A \xrightarrow{\delta} \Sigma A \xrightarrow{-\Sigma j} \dots \longrightarrow \Sigma^n A \xrightarrow{(-1)^n \Sigma^n j} \Sigma^n X \longrightarrow \dots \tag{3.20}$$

Each pair of successive maps in this sequence is null homotopic, and each three term sequence is pointwise equivalent to a strict cofiber sequence as in (3.14).

Exercise 3.7 Show that for any space Y there is a long exact sequence

$$[A,Y] \xleftarrow{j^*} [X,Y] \xleftarrow{q^*} [X/A,Y] \xleftarrow{\delta^*} [\Sigma A,Y] \xleftarrow{-\Sigma j^*} [\Sigma X,Y] \xleftarrow{-\Sigma q^*} [\Sigma(X/A),Y] \longleftarrow \dots$$

$$\dots \longleftarrow [\Sigma^n A,Y] \xleftarrow{(-1)^n \Sigma j^*} [\Sigma^n X,Y] \xleftarrow{(-1)^n \Sigma^n q^*} [\Sigma^n(X/A),Y] \longleftarrow \dots$$

in which the first three terms are exact in the sense of pointed sets, the next three terms are exact in the sense of groups, and the remaining terms are exact in the sense of abelian groups. □

4 Applications

There are no exercises in this section. Instead you will find here a collection of examples to illustrate how the machinery you have developed may be applied.

Example 4.1 *Let $j : A \hookrightarrow X$ be a cofibration. Assume that $j \simeq *$. Then according to Exercise 3.1 of the* Cofibrations *sheet, X retracts off of X/A up to homotopy. Now with a little bit more technology available we can offer an improvement on this statement.*

Consider the mapping cone

$$\begin{tikzcd} A \arrow[r, "in_0"] \arrow[d, "j"'] \arrow[dr, phantom, "\lrcorner", very near end] & CA \arrow[d] \\ X \arrow[r, "i_j"'] & C_j. \end{tikzcd} \tag{4.1}$$

The map $A \hookrightarrow CA$ is a cofibration, so according to Exercise 3.2 of Cofibrations *the pushout space C_j is homotopy equivalent to the pushout of $X \xleftarrow{*} A \xrightarrow{in_0} CA$. On the other hand, by Exercise 3.4 above, there is a homotopy equivalence $C_j \xrightarrow{\simeq} C_j/CA \cong X/A$. In particular*

$$X/A \simeq X \vee \Sigma A. \tag{4.2}$$

□

Example 4.2 *We start with the cofibration $1 = id_{S^0} : S^0 \xrightarrow{=} S^0$ and get from it the long cofiber sequence*

$$S^0 \xrightarrow{1} S^0 \to * \to S^1 \xrightarrow{-1} S^1 \to * \dots \tag{4.3}$$

Our purpose here is to verify that the signs of Exercise 3.6 are correct.

We check that the mapping cone of 1 is $CS^0 \cong D^1$, and that the inclusion i_1 is the standard map. Then the mapping cone of i_1 is S^1, but now $i_{i_1} : D^1 \hookrightarrow S^1$ is the inclusion of the bottom *cone. i.e. the southern hemisphere. This means that in the diagram*

$$\begin{tikzcd} & & D^1 \arrow[r, "i_{i_1}"] \arrow[d, equal] & S^1 \arrow[r, "q_{i_j}"] \arrow[d, "\simeq"'] & S^1 \\ & S^0 \arrow[r, "i_1"] \arrow[d, equal] & D^1 \arrow[r, "q_j"] \arrow[d, "\simeq"'] & S^1 \arrow[ur, dashed] & \\ S^0 \arrow[r, "1"] & S^0 \arrow[r] & *. \arrow[ur, dashed, "\delta"'] & & \end{tikzcd} \tag{4.4}$$

(compare (3.19)) the vertical homotopy equivalence $S^1 \xrightarrow{\simeq} S^1$ must interchange the northern and southern hemispheres. Thus the unlabelled dotted arrow is indeed of degree -1. $\square$

Example 4.3 *Suppose X, Y are your favourite well-pointed spaces. Under these assumptions on X, Y, the inclusion $j : X \vee Y \hookrightarrow X \times Y$ is a cofibration, and $(X \times Y)/(X \vee Y) = X \wedge Y$ by definition. Thus there is a long cofiber sequence*

$$X \vee Y \to X \times Y \to X \wedge Y \xrightarrow{\delta} \Sigma X \vee \Sigma Y \xrightarrow{\Sigma j} \Sigma(X \times Y) \to \Sigma(X \wedge Y) \to \dots \tag{4.5}$$

(we don't need be pedantic about signs here). We claim that Σj has a left homotopy inverse. Indeed, denote by

$$X \xleftarrow{pr_X} X \times Y \xrightarrow{pr_Y} Y \tag{4.6}$$

the projections and by

$$X \xrightarrow{i_X} X \vee Y \xleftarrow{i_Y} Y \tag{4.7}$$

the inclusions. Now use the suspension comultiplication to form the map

$$\alpha = \Sigma(i_X pr_X) + \Sigma(i_Y pr_Y) : \Sigma(X \times Y) \to \Sigma X \vee \Sigma Y. \tag{4.8}$$

Then

$$\begin{aligned} \Sigma j^* \alpha &= (\Sigma(i_X pr_X) + \Sigma(i_Y pr_Y))\, \Sigma j \\ &\simeq \Sigma(i_X pr_X) + \Sigma(i_Y pr_Y j) \\ &\simeq \Sigma(i_X q_X) + \Sigma(i_Y q_Y) \\ &\simeq id \end{aligned} \tag{4.9}$$

where

$$X \xleftarrow{q_X} X \vee Y \xrightarrow{q_Y} Y \tag{4.10}$$

are the pinch maps.

Then a consequence of (4.9) is that the connecting map δ in (4.5) is null homotopic. For

$$\delta \simeq id \circ \delta \simeq (\alpha \Sigma j)\delta = \alpha(\Sigma j \delta) \simeq \alpha * = * \tag{4.11}$$

since Σj and δ are consecutive maps in a cofiber sequence. Now repeat the analysis of Example 4.1, replacing $\Sigma(X \times Y)$ up to homotopy with the mapping cone C_δ. The result is a homotopy equivalence

$$\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y). \tag{4.12}$$

$\square$

Example 4.4 *In Exercise Sheet 3: Co-H-Spaces we defined homotopy groups with coefficients. Recall that the n-dimensional Moore space of degree k is the space $P^n(k) = S^{n-1} \cup_k e^n$ which is the mapping cone of the degree k map $k : S^{n-1} \to S^{n-1}$*

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner"] & D^n \arrow[d] \\ S^{n-1} \arrow[r] & P^n(k) \end{tikzcd} \qquad n \geq 3,\ k \geq 2. \tag{4.13}$$

In particular there is a cofiber sequence

$$\ldots \to P^{n-1}(k) \to S^{n-1} \xrightarrow{k} S^{n-1} \to P^n(k) \to S^n \xrightarrow{k} S^n \to P^{n+1}(k) \to \ldots \tag{4.14}$$

so given a space X we get a long exact sequence of groups

$$\ldots \pi_{n-1}X \xleftarrow{k} \pi_{n-1}X \leftarrow \pi_n(X;\mathbb{Z}_k) \leftarrow \pi_n X \xleftarrow{k} \pi_n X \leftarrow \ldots \tag{4.15}$$

Since we are assuming $n \geq 3$, the groups $\pi_{n-1}X$ and $\pi_n X$ are abelian. If $n \geq 4$, then $\pi_n(X;\mathbb{Z}_k)$ is abelian too. Note that the induced maps $k : \pi_r X \to \pi_r X$ are multiplication by k.

Now from (4.15) *we extract the short exact sequence*

$$0 \leftarrow \ker\left(\pi_{n-1}X \xleftarrow{k} \pi_{n-1}X\right) \leftarrow \pi_n(X;\mathbb{Z}_k) \leftarrow \frac{\pi_n X}{k \cdot \pi_n X} \leftarrow 0. \tag{4.16}$$

We identify

$$\frac{\pi_n X}{k \cdot \pi_n X} \cong \pi_n X \otimes \mathbb{Z}_k \tag{4.17}$$

and

$$\ker\left(\pi_{n-1}X \xleftarrow{k} \pi_{n-1}X\right) \cong Tor^{\mathbb{Z}}(\pi_{n-1}X, \mathbb{Z}_k) \tag{4.18}$$

and so turn (4.16) *into the* ***universal coefficient exact sequence***

$$0 \to Tor^{\mathbb{Z}}(\pi_{n-1}X, \mathbb{Z}_k) \to \pi_n(X;\mathbb{Z}_k) \to \pi_n X \otimes \mathbb{Z}_k \to 0. \tag{4.19}$$

□

References

[1] P. May, K. Ponto, *More Concise Algebraic Topology*, University of Chicago Press, (2012).

[2] A. Strøm. *The Homotopy Category is a Homotopy Category*, Arch. Math. (Basel), **23**, (1972), 435-441.